C3 S14 ve

1. The curve C has equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = \frac{4x+1}{x-2}, \quad x > 2$$

(a) Show that

$$\mathrm{f}'(x) = \frac{-9}{(x-2)^2}$$

(3)

Given that P is a point on C such that $\mathrm{f}'(x) = -1$,

(b) find the coordinates of P.

(3)

a) $u = 4x+1 \qquad v = x-2 \qquad f'(x) = \dfrac{4(x-2)-(4x+1)}{(x-2)^2}$

$u' = 4 \qquad v' = 1$

$= \dfrac{4x-8-4x-1}{(x-2)^2} = \dfrac{-9}{(x-2)^2}$ #

b) $f'(x) = -1 \Rightarrow (x-2)^2 = 9 \qquad x = 2 \pm 3 \quad \therefore x = 5$

$f(5) = \dfrac{21}{3} = 7 \qquad P(5, 7)$

2. Find the exact solutions, in their simplest form, to the equations

(a) $2\ln(2x+1) - 10 = 0$

(2)

(b) $3^x e^{4x} = e^7$

(4)

a) $\ln(2x+1) = 5$

$2x+1 = e^5 \quad \therefore \quad x = \dfrac{-1+e^5}{2}$

b) $3^x = \dfrac{e^7}{e^{4x}} \Rightarrow 3^x = e^{7-4x}$

$\ln 3^x = 7 - 4x$

$4x + x\ln 3 = 7$

$x(4+\ln 3) = 7 \quad \therefore \quad x = \dfrac{7}{4+\ln 3}$

3. The curve C has equation $x = 8y \tan 2y$

The point P has coordinates $\left(\pi, \frac{\pi}{8}\right)$

(a) Verify that P lies on C.

(1)

(b) Find the equation of the tangent to C at P in the form $ay = x + b$, where the constants a and b are to be found in terms of π.

(7)

a) $y = \frac{\pi}{8}$ $\quad x = 8\left(\frac{\pi}{8}\right)\tan\left(\frac{\pi}{4}\right) = \pi \times 1 = \pi$ #

b) $u = 8y \quad v = \tan 2y$

$u' = 8 \quad v' = 2\sec^2 2y$

$$\frac{dx}{dy} = 8\tan 2y + 16y\sec^2 2y$$

$$\therefore \frac{dy}{dx} = \frac{1}{8\tan 2y + 16y\sec^2 2y}$$

$$y = \frac{\pi}{8} \quad M_t = \frac{1}{8\tan\left(\frac{\pi}{4}\right) + \frac{2\pi}{\left(\cos\frac{\pi}{4}\right)^2}} = \frac{1}{8 + 4\pi}$$

$$y - \frac{\pi}{8} = \frac{1}{(8+4\pi)}(x - \pi) \Rightarrow (8+4\pi)y - \frac{\pi}{8}(8+4\pi) = x - \pi$$

$$\Rightarrow (8+4\pi)y = x - \cancel{\pi} + \cancel{\pi} + \frac{\pi^2}{2}$$

$$4(2+\pi)y = x - \frac{\pi^2}{2}$$

4.

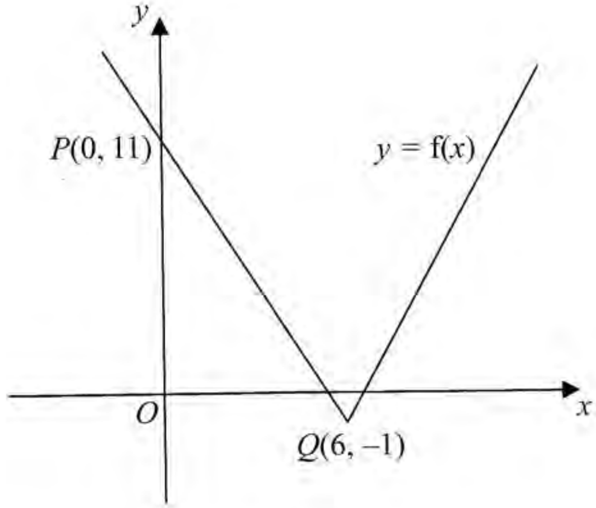


Figure 1

Figure 1 shows part of the graph with equation $y = \mathrm{f}(x)$, $x \in \mathbb{R}$.

The graph consists of two line segments that meet at the point $Q(6, -1)$.

The graph crosses the y-axis at the point $P(0, 11)$.

Sketch, on separate diagrams, the graphs of

(a) $y = |\mathrm{f}(x)|$ **(2)**

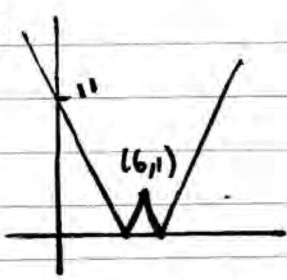
(b) $y = 2\mathrm{f}(-x) + 3$ **(3)**

On each diagram, show the coordinates of the points corresponding to P and Q.

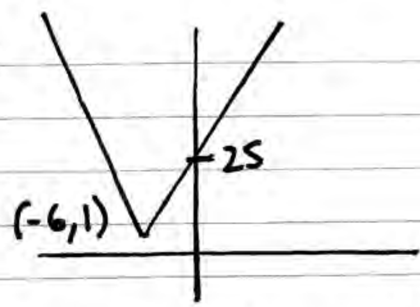
Given that $\mathrm{f}(x) = a|x - b| - 1$, where a and b are constants,

(c) state the value of a and the value of b. **(2)**

a)



b)



c) $x = 6,\ f(x) = -1$ $\quad a|6-b| - 1 = -1$ $\quad \Rightarrow a|6-b| = 0$ $\quad a = 0, b = 6$

$x = 0\ f(x) = 11$ $\quad a|-b| - 1 = 11$ $\quad a \times \pm b = 12$ $\quad a \neq 0$

$b = 6 \quad a = 2$

5.

$$g(x) = \frac{x}{x+3} + \frac{3(2x+1)}{x^2+x-6}, \quad x > 3$$

(a) Show that $g(x) = \dfrac{x+1}{x-2}, \quad x > 3$ **(4)**

(b) Find the range of g. **(2)**

(c) Find the exact value of a for which $g(a) = g^{-1}(a)$. **(4)**

a) $g(x) = \dfrac{x}{x+3} + \dfrac{3(2x+1)}{(x+3)(x-2)} = \dfrac{x(x-2)+3(2x+1)}{(x+3)(x-2)}$

$g(x) = \dfrac{x^2-2x+6x+3}{(x+3)(x-2)} = \dfrac{x^2+4x+3}{(x+3)(x-2)} = \dfrac{\cancel{(x+3)}(x+1)}{\cancel{(x+3)}(x-2)}$ #

b) $g(x) > 1 \quad \dfrac{3+1}{3-2} = \dfrac{4}{1} = 4 \quad \therefore \quad 1 < g(x) < 4$

c) $x = \dfrac{y+1}{y-2} \Rightarrow xy - 2x = y+1 \Rightarrow xy - y = 1 + 2x$

$\therefore y = \dfrac{1+2x}{x-1} = g^{-1}(x)$

$\dfrac{a+1}{a-2} = \dfrac{1+2a}{a-1} \Rightarrow a^2 - 1 = 2a^2 - 3a - 2$

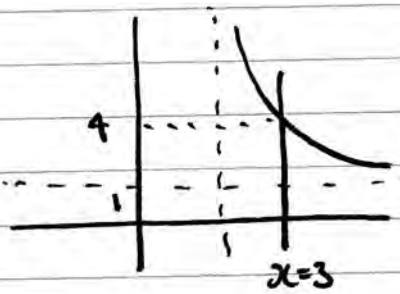
$\Rightarrow a^2 - 3a - 1 = 0$

$(a - \frac{3}{2})^2 - \frac{9}{4} = 1$

$(a - \frac{3}{2})^2 = \frac{13}{4}$

$a = \frac{3}{2} + \frac{\sqrt{13}}{2}$

$\dfrac{x+1}{x-2} \quad x \neq 2$



range $1 < y < 4$

6.

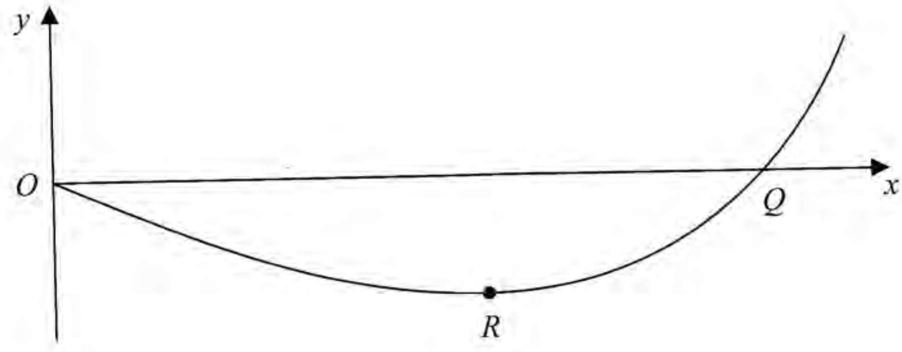


Figure 2

Figure 2 shows a sketch of part of the curve with equation

$$y = 2\cos\left(\frac{1}{2}x^2\right) + x^3 - 3x - 2$$

The curve crosses the x-axis at the point Q and has a minimum turning point at R.

(a) Show that the x coordinate of Q lies between 2.1 and 2.2 **(2)**

(b) Show that the x coordinate of R is a solution of the equation

$$x = \sqrt{1 + \frac{2}{3}x\sin\left(\frac{1}{2}x^2\right)}$$

(4)

Using the iterative formula

$$x_{n+1} = \sqrt{1 + \frac{2}{3}x_n\sin\left(\frac{1}{2}x_n^{\,2}\right)}, \quad x_0 = 1.3$$

(c) find the values of x_1 and x_2 to 3 decimal places. **(2)**

a) $x = 2.1 \quad y = -0.22$ $\therefore$ by sign change rule
$x = 2.2 \quad y = 0.55$ $\quad 2.1 < x < 2.2$

b) $x = \sqrt{1 + \frac{2}{3}x\sin\left(\frac{1}{2}x^2\right)}$ when $\frac{dy}{dx} = 0$ TP.

$\frac{dy}{dx} = -2x\sin\left(\frac{x^2}{2}\right) + 3x^2 - 3$ $\therefore$ at TP $3x^2 = 3 + 2x\sin\left(\frac{x^2}{2}\right)$

$\therefore x^2 = 1 + \frac{2}{3}x\sin\left(\frac{x^2}{2}\right)$

c) $x_0 = 1.3 \quad x_1 = 1.284 \quad x_2 = 1.276$

$\therefore x = \sqrt{1 + \frac{2}{3}x\sin\left(\frac{x^2}{2}\right)}$

7. (a) Show that

$$\operatorname{cosec} 2x + \cot 2x = \cot x, \qquad x \neq 90n°, \qquad n \in \mathbb{Z}$$

(5)

(b) Hence, or otherwise, solve, for $0 \leqslant \theta < 180°$,

$$\operatorname{cosec}(4\theta + 10°) + \cot(4\theta + 10°) = \sqrt{3}$$

You must show your working.

(Solutions based entirely on graphical or numerical methods are not acceptable.)

(5)

a) $\frac{1}{\sin 2x} + \frac{\cos 2x}{\sin 2x} =$

$= \frac{1 + (2\cos^2 x - 1)}{2\sin x \cos x} = \frac{\cancel{2}\cos^{\cancel{2}} x}{\cancel{2}\sin x \cancel{\cos x}} = \frac{\cos x}{\sin x} = \cot x$

b) $\operatorname{Cosec} 2x + \cot 2x = \cot x$

$\operatorname{Cosec}(4\theta+10) + \cot(4\theta+10) = \sqrt{3}$

$\therefore \cot x = \sqrt{3} \Rightarrow \tan x = \frac{1}{\sqrt{3}} \quad \therefore x = 30, 210, 390, 570$

$2x = 4\theta + 10 = 60, 420, 780, 1040$

$4\theta = \quad = 50, 410, 770, 1030$

$\therefore \theta = 12.5°, 102.5°$

8. A rare species of primrose is being studied. The population, P, of primroses at time t years after the study started is modelled by the equation

$$P = \frac{800e^{0.1t}}{1+3e^{0.1t}}, \qquad t \geqslant 0, \quad t \in \mathbb{R}$$

(a) Calculate the number of primroses at the start of the study.
(2)

(b) Find the exact value of t when $P = 250$, giving your answer in the form $a \ln(b)$ where a and b are integers.
(4)

(c) Find the exact value of $\frac{dP}{dt}$ when $t = 10$. Give your answer in its simplest form.
(4)

(d) Explain why the population of primroses can never be 270
(1)

a) $t=0 \quad P = \frac{800}{1+3} = \underline{200}$

b) $250 = \frac{800e^{0.1t}}{1+3e^{0.1t}} \Rightarrow 800e^{0.1t} = 250 + 750e^{0.1t}$

$\Rightarrow 50e^{0.1t} = 250 \Rightarrow e^{0.1t} = 5 \Rightarrow 0.1t = \ln 5$

$\therefore t = \underline{10\ln 5}$

c) $u = 800e^{0.1t} \quad v = 1+3e^{0.1t}$

$u' = 80e^{0.1t} \quad v' = 0.3e^{0.1t}$

$= \frac{(1+3e^{0.1t})80e^{0.1t} - (800e^{0.1t})(0.3e^{0.1t})}{(1+3e^{0.1t})^2}$

$t=10 \quad \frac{80e + 240e^2 - 240e^2}{(1+3e)^2} = \frac{80e}{(1+3e)^2}$

d) $P = \dfrac{800e^{0.1t}}{1+3e^{0.1t}} \begin{matrix} \div e^{0.1t} \\ \div e^{0.1t} \end{matrix} = \dfrac{800}{e^{-0.1t}+3}$

as $t \to \infty$ $\quad P \to \dfrac{800}{3} \to 266.\dot{6}$

$\therefore$ Population can never reach 270

9. (a) Express $2\sin\theta - 4\cos\theta$ in the form $R\sin(\theta - \alpha)$, where R and α are constants, $R > 0$ and $0 < \alpha < \frac{\pi}{2}$

Give the value of α to 3 decimal places.

(3)

$$H(\theta) = 4 + 5(2\sin 3\theta - 4\cos 3\theta)^2$$

Find

(b) (i) the maximum value of $H(\theta)$,

(ii) the smallest value of θ, for $0 \leqslant \theta < \pi$, at which this maximum value occurs.

(3)

Find

(c) (i) the minimum value of $H(\theta)$,

(ii) the largest value of θ, for $0 \leqslant \theta < \pi$, at which this minimum value occurs.

(3)

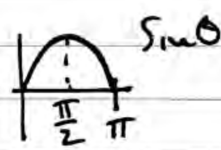
$R\sin(\theta - \alpha) = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$

$\quad\quad 2\sin\theta \quad - 4\cos\theta$

$\frac{R\sin\alpha = 4}{R\cos\alpha = 2} \Rightarrow \tan\alpha = 2 \quad \therefore \alpha = 1.107^c$

$R = \sqrt{4^2 + 2^2} \quad R = 2\sqrt{5}$

$\therefore 2\sqrt{5}\sin(\theta - 1.107)$



b) $\text{Max}(2\sin 3\theta - 4\cos 3\theta) = 2\sqrt{5}$

when $3\theta - 1.107 = \frac{\pi}{2}$ $\theta = 0.89264..$

$H_{max} = 4 + 5(2\sqrt{5})^2 = 4 + 100 = 104$ when $\theta = 0.893$

c) $H_{min} = 4 + 5(0) = 4$ $\quad\quad 0 \leq \pi < \pi$

when $3\theta - 1.107 = 0, \pi, 2\pi, ...$

$3\theta = 1.107, \pi + 1.107, 2\pi + 1.107, 3\pi + 1.107$

$\therefore \max\theta = 2.463$